Modeling Dividends and Other Distributions
Reconciling The Change In Asset Value Over Time

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In this white paper we will build a model to reconcile the cumulative change in random asset value over the time interval $[0,t]$. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with reconciling the change in asset value over the time interval $[0,5]$ for each random draw from a normal distribution. We are given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Description	Value
Asset value at time zero ($)	1,000,000
Expected return - mean (%)	10.00
Expected return - volatility (%)	20.00
Dividends and other distributions (%)	4.00

Our task is to answer the following question given that the random draws from a normal distribution with mean zero and variance one are 2.00, 1.00, 0.00, -1.00 and -2.00.

Question: Reconcile the change in asset value for each random draw above.

Modeling Asset Value Over Time

We will define the variable A_t to be asset value at time t, the variable μ to be the expected rate of return, the variable ϕ to be the dividend yield, the variable σ to be expected return volatility, and the variable δW_t to be the change in the underlying brownian motion at time t. The stochastic differential equation for the change in asset value over the time interval $[t, t+\delta t]$ is...

$$\delta A_t = \mu\, A_t\, \delta t - \phi\, A_t\, \delta t + \sigma\, A_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\left[0, \delta t\right] \tag{1}$$

The solution to the SDE in Equation (1) above is the equation for random asset value at time t, which is...

$$A_t = A_0 \operatorname{Exp}\left\{\left(\mu - \phi - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}\, z\right\} \text{ ...where... } z \sim N\left[0, 1\right] \tag{2}$$

We are currently standing at time zero and want to simulate asset prices at time t. We will define the variable $z(n)$ to be the n'th random variate pulled from a normal distribution with mean zero and variance one. If we define the variable $\theta(n)$ to random asset return for the n'th trial then the equation for random return over the time interval $[0,t]$ is...

$$\theta(n) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}\, z(n) \tag{3}$$

Using Equations (2) and (3) above the equation for random asset value at time t for the n'th trial is...

$$A(n)_t = A_0 \operatorname{Exp}\left\{\theta(n) - \phi\, t\right\} \tag{4}$$

Using Equation (4) above the equation for asset value at any time $0 \leq s \leq t$ is...

$$A(n)_s = A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, s\right\} \text{ ...where... } \lambda = \frac{\theta(n)}{t} \tag{5}$$

The derivative of Equation (5) above with respect to the time variable s is...

$$\frac{\delta A(n)_s}{\delta s} = \lambda\, A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, s\right\} - \phi\, A_0 \,\mathrm{Exp}\left\{\frac{\theta(n)}{t}\, s - \phi\, s\right\} \tag{6}$$

Note that we can rewrite Equation (6) above as...

$$\delta A(n)_s = \lambda\, A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, s\right\} \delta s - \phi\, A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, s\right\} \delta s \tag{7}$$

Using the first half of Equation (7) above the equation for total return over the time interval $[0, t]$ is...

$$\text{Total return} = \int\limits_0^t \lambda\, A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, u\right\} \delta u = \frac{\lambda\, A_0}{\lambda - \phi} \,\mathrm{Exp}\left\{(\lambda - \phi)\, u\right\} \Bigg[_{u=0}^{u=t} = \frac{\lambda\, A_0}{\lambda - \phi} \left(\mathrm{Exp}\left\{(\lambda - \phi)\, t\right\} - 1\right) \tag{8}$$

Using the second half of Equation (7) above the equation for total dividends paid out over the time interval $[0, t]$ is...

$$\text{Total dividends} = \int\limits_0^t \phi\, A_0 \,\mathrm{Exp}\left\{(\lambda - \phi)\, u\right\} \delta u = \frac{\phi\, A_0}{\lambda - \phi} \,\mathrm{Exp}\left\{(\lambda - \phi)\, u\right\} \Bigg[_{u=0}^{u=t} = \frac{\phi\, A_0}{\lambda - \phi} \left(\mathrm{Exp}\left\{(\lambda - \phi)\, t\right\} - 1\right) \tag{9}$$

The Answer To Our Hypothetical Problem

Using the model parameters in Table 1 above and the equations above the answer to the question is...

Table 2: Asset Value Reconciliation

Random Draw	Beginning Value	Total Return	Dividends Paid	Ending Value
2.00	1,000,000	2,281,672	-352,047	2,929,625
1.00	1,000,000	1,146,050	-272,826	1,873,224
0.00	1,000,000	412,664	-214,911	1,197,753
-1.00	1,000,000	-62,023	-172,125	765,852
-2.00	1,000,000	-370,146	-140,163	489,691